Blending Type Bernstein Operators

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ABSTRACT

The aim of my thesis is to examine the Bernstein Operators, which are linear positive operators and the properties of the New Generalized Operators.

My thesis consists of four parts. The first part is an introduction and gives information about the parts that we will examine in the following chapters.

The second chapter is to give more information about all the fundamental theorems and properties. In this section, the basic theorems used in the thesis are proved and explained with examples.

In the third part, Korovkin Theorem's proof and Bernstein Operators and approximation propeties and converges uniformly of Bernstein operators are given.

In the last, new family of generalized Bernstein operators' definition and some important theory of convergence approximation of functions are given. After that we will examine some important results regarding the rate of converges and predictions of new generelized operators, that are appliactions of the properties and formulas which are foretold. Lastly, we examine the shape preservation properties and complete the thesis.

Keywords: Modulus of Continuity, Rate of Converges, Lipschitz function, Korovkin Theorem, Bernstein Operators, Shape Preserving.

ÖZ

Tezimin amacı lineer pozitif operatörler olan Bernstein Operatörlerini ve yeni genelleştirilmiş operatörlerin özelliklerini incelemektedir.

Bu tez dört üniteden oluşmaktadır. Birinci ünite giriştir ve bundan sonraki üç ünitede kullanılacak olan temel kavramları açıklar.

İkinci ünite, tüm temel teoremler ve özellikler hakkında daha fazla bilgi vermektir. Bu bölümde tezde kullanılan temel teoremler ispatlanmıştır ve örneklerle açıklanmıştır.

Üçüncü ünitede Korovkin Teoremi ve Bernstein Operatörlerinin ispatı ve Bernstein operatörlerinin yaklaşım özellikleri ve hızları verilmiştir.

Son ünitede, genelleştirilmiş Bernstein Operatörlerinin yeni bir ailesinin tanımını ve fonksiyonların düzgün yakınsama yaklaşımı teorisinde önemli bir rol oynayan bazı temel özellikleri veriyoruz. Daha sonra, ilk bölümde hatırlatılan özelliklerin ve formüllerin doğrudan uygulamaları olan yeni doğrusal pozitif operatörlerin yakınsama oranları ve tahminleriyle ilgili bazı önemli sonuçları inceleyeceğiz. Son olarak, yeni operatörün şekil koruma özelliklerini inceliyoruz ve tezi tamamlıyoruz.

Anahtar Kelimeler: Süreklilik modülü, yakınsama oranı, Lipschitz Fonksiyonu, Korovkin Teoremi, Bernstein operatörleri, şekil koruma özellikleri.

I dedicate this thesis to my precious family, who has always supported me and raised me to this day and to the mathematics department of Eastern Mediterranean University, which is a second family to me.

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Chapter 1

INTRODUCTION

The main subject of my thesis is to study Blending type Bernstein operators and the approximation properties of a class of these operators. In approximation theory, which is a very important area of mathematical analysis; it is intended to obtain a representation of an arbitrary function, means other simple and more useful functions. For this, first of all, introductory information and some basic concepts are given in the second chapter. This general information section includes what is a linear positive operators, basic properties and concepts of the theory of approximation, the modulus of continuity and their properties. Most of the concepts are supported with examples for better understanding of the subject. As we finish the second chapter, all the terms that we will use in the third and fourth chapters are explained.

The third part started with the proof of Korovkin theorem. In 1885, the German mathematician Weierstrass proved the existence of at least one polynomial that converges to all continuous function on a finite interval. The difference of the theorem of Weierstrass approximation from Taylor's theorem, which is expressed as a function with enough derivatives, locally approximated by Taylor polynomials is that it is applied to a continuous function which isn't necessarily differentiable and there is a polynomial that converges to this function over the [a,b]. Using this theorem of Weierstrass, Bernstein showed that Bernstein polynomials on [0,1] for the arbitrary function f is continuous on [0,1]. Studies of Bernstein polynomials contributed to the formation of the theory of linear positive operators in approximation problems.

Finally, in the fourth part, we will explain the new generalized Bernstein operators. For this, we will explain the monotonic and convex properties, shape preservation. The basic aim in this chapter is examine some important results regarding the rate of converges and predictions of the new generalized operators.

Chapter 2

PRELIMNIARIES

This chapter is devoted to the basic concepts and facts that are needed for the all thesis. The present thesis is related with positive linear operators and these operators are based on functions, function spaces and some special definitions and theorems. You can find the basic definitions and properties related with all of these.

2.1 Space of the Continuous Function in a Finite Range

In the section we introduce and discuss the main definitions and properties about the concept of continuous functions. Some definitions are illusrated by examples for the better understanding of readers.

Definition 2.1.1: Assume $h(\varkappa)$ is a function defined on a domain which includes the point $\varkappa_0$. A function $h(\varkappa)$ is continuous at $\varkappa = \varkappa_0$, if it satisfies the following conditions:

i. $h(\varkappa_0)$ exists.

ii. $\lim_{\varkappa \to \varkappa_0} h(\varkappa)$ exists.

iii. $\lim_{\varkappa \to \varkappa_0} h(\varkappa) = h(\varkappa_0)$.

Example 2.1.1: $h(\varkappa) = |\varkappa| = \begin{cases} \varkappa, & \text{if } \varkappa \geq 0 \\ -\varkappa, & \text{if } \varkappa < 0 \end{cases}$

i. $h(0) = 0$, so it is defined at $\varkappa = 0$.

ii. $\lim_{\varkappa \to 0} h(\varkappa)$=0, so limit exists.

iii. $\lim_{\varkappa \to 0} h(\varkappa) = h(\varkappa_0) = 0.$

The given function is continuous as it satisfies all conditions.

Example 2.1.2: $h(\varkappa) = \begin{cases} 2\varkappa + 1, & \text{if } \varkappa < 1 \\ 2, & \text{if } \varkappa = 1 \\ -\varkappa + 4, & \text{if } \varkappa > 1 \end{cases}$

i. $h(1) = 2$, so it is defined at $\varkappa = 1$.

ii. $\lim_{\varkappa \to 1} h(\varkappa)$=3, so limit exists.

iii. $\lim_{\varkappa \to 1} h(\varkappa) \neq h(1)$.

Therefore, $h(\varkappa)$ isn't continuous at $\varkappa = 1$ since $\lim_{\varkappa \to 1} h(\varkappa) \neq h(1)$.

Definition 2.1.2: Assume E is a non-empty subset of $\mathbb{R}$ and $\{h_m\}$ is a real valued functions where $m \geq 1$ and h described on E. $\{h_m\}$ is called pointwise convergent to h on E if $\forall\, x \in$ E the sequence $\{h_m(x)\}$ of real numbers converges to the function $h(\varkappa)$.

It means, $\forall\, \varkappa \in E$, $\exists\, \varepsilon > 0$ and $\mathrm{m}_0(\varkappa)$ such that;

$$|h_{\mathrm{m}}(\varkappa) - h(\varkappa)| < \varepsilon, \qquad \forall\, \mathrm{m} \geq \mathrm{m}_0(\varkappa).$$

The notation $h_m \to h$ is used to denote the pointwise convergence of $\{h_m\}$ to h.

Definition 2.1.3: Assume E is a non-empty subset of $\mathbb{R}$ and $\{h_m\}$ is a real valued functions where $m \geq 1$ and h described on E and m is independent from $\varkappa$. $\{h_m\}$ is called uniformly convergent to h, if $\varepsilon > 0$, $\exists\, m_0$ such that;

$$|h_m(\varkappa) - h(\varkappa)| < \varepsilon, \qquad \forall\, m \geq m_0\,, \varkappa \in E.$$

The notation $h_m \rightrightarrows h$ is used to denote the uniform convergence of $\{h_m\}$ to h.

Example 2.1.3: For any $\varkappa \in \mathbb{R}$

$$h_{\mathrm{m}}(\varkappa) = \frac{1}{\mathrm{m}(1+\varkappa^2)}$$

is given.

Since $\frac{1}{(1+\varkappa^2)} \leq 1, \forall\, \varkappa \in \mathbb{R}$ we have;

$$|\, h_{\mathrm{m}}(\varkappa) - h(\varkappa)| = \left|\frac{1}{\mathrm{m}(1+\varkappa^2)} - 0\right| \leq \frac{1}{\mathrm{m}}.$$

So, for a given $\varepsilon > 0$, choose $\mathrm{N} > \frac{1}{\varepsilon}$, which is independent on $\varkappa \in \mathbb{R}$, then

$|\, h_{\mathrm{m}}(\varkappa) - h(\varkappa)| < \varepsilon,$

$\forall\, \varkappa \in \mathbb{R}.$ Therefore, $h_{\mathrm{m}} \rightrightarrows h.$

2.2 Linear Space

In mathematics, linear spaces are used for different purposes. Mostly it is used within functional analysis. In this section, we will examine linear spaces in detail.

Definition 2.2.1: Assume X is a set and K be a scalar field. If two operators addition and scalar multiplication

$+ : XxX \to X, (\varkappa, y) \to \varkappa + y,$

$\cdot : KxX \to X, (\gamma, \varkappa) \to \gamma\varkappa$, satisfies the following conditions;

$\forall\, \varkappa, \mathrm{y}, \mathrm{z} \in \mathrm{X}$ and $\gamma, \delta \in \mathrm{K}$,

i. $(\varkappa + \mathrm{y}) + \mathrm{z} = \varkappa + (\mathrm{y} + \mathrm{z}),$

ii. $\varkappa + \mathrm{y} = \mathrm{y} + \varkappa,$

iii. $\forall\, \varkappa \in \mathrm{X}, \exists\, 0 \in \mathrm{X}$ such that $\varkappa + 0 = 0 + \varkappa = \varkappa,$

iv. $\forall\, \varkappa \in \mathrm{X}$ and $\varkappa \neq 0, \exists\, (-\varkappa) \in \mathrm{X}$ such that $\varkappa + (-\varkappa) = (-\varkappa) + \varkappa = 0$

v. $\gamma(\varkappa + \mathrm{y}) = \gamma\varkappa + \gamma\mathrm{y},$

vi. $(\gamma + \delta)\varkappa = \gamma\varkappa + \delta\varkappa,$

vii. $\gamma(\delta\varkappa) = (\gamma\delta)\varkappa$,

viii. $1.\varkappa = \varkappa$.

Then X is called linear space over the scalar field K. If $K = \mathbb{R}$, then it is called real vector space.

Definition 2.2.2: Assume X is a vector space. A function with real value,

$$\varkappa \rightarrow \|\varkappa\|$$

for $\varkappa, y \in X$ and $\gamma \in M$ is called a norm over X, if it satifies;

i. $\|\varkappa\| \geq 0$ and $\|\varkappa\| = 0 \Leftrightarrow x = 0$.

ii. $\|\gamma\varkappa\| = |\gamma|\|\varkappa\|$.

iii. $\|\varkappa + y\| \leq \|\varkappa\| + \|y\|$.

Example 2.2.1: Let $|\varkappa|$ denote the absolute value of each $\varkappa \in \mathbb{R}$. The set of $\mathbb{R}$ is a vector space on $\mathbb{R}$ and function

$$\varkappa \rightarrow |\varkappa|$$

is a norm on the vector space $\mathbb{R}$.

Example 2.2.2: For all $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_n) \in \mathbb{R}^n$, this transformation,

$$\varkappa \rightarrow \|\varkappa\|_{max} = \max\{\, |\varkappa_i| : (1 \leq i \leq n) \,\}$$

is a norm above $\mathbb{R}^n$.

Definition 2.2.3: Let X = C[a, b] be the set of all continuous functions defined on [a,b]. Then C[a, b] is defined with following operations:

$\forall\ f, g \in C[a, b]$ and $\forall\ \lambda \in \mathbb{R}$

i. $(f + g)(k) = f(k) + g(k)$.

ii. $f(\lambda k) = \lambda f(k)$.

Therefore, the vector space of C[a, b] is called space of continuous function.

Definition 2.2.4: The linear space X, which is defined norm on $\|.\|$, is called the normed vector space and shown as $(X, \|.\|)$.

Example 2.2.5: Let C[a,b] denotes the space of continuous function space which are continous on [a,b]. Since [a,b] is compact and f is continuous it takes the maximum and the minimum values. Therefore we can defined the norm;

$$\|f\|_C = \max_{a \le \varkappa \le b} |f(\varkappa)|, \quad \text{on } C[a, b].$$

Definition 2.2.5: Assume that $f \in C[a, b]$ and f_m is a sequence of functions in C[a,b] then, the f_m converge uniformly to the function f on $[a, b]$ if and only if it satisfies; $\forall\, \varkappa \in [a, b]$

$$|\, f_m(\varkappa) - f(\varkappa)| < M\varepsilon_n$$

where ε_m is a sequence which converge to zero and $M > 0$. This convergence is denoted by;

$$f_m(\varkappa) \rightrightarrows f(\varkappa).$$

2.3 Operators

An operator is a special kind of function. Operators take a function as an input and give a function as an output. Therfore operators are more general objects than functions. In this section we will examine operators, which are linear and positive and some of their basic properties that are needed later.

Definition 2.3.1: Assume X and Y are function spaces. If there is a function L that corresponds to a function g in Y space for any function f taken from X, it is called an operator on X and for each $f \in X$ it is denoted as

$$L(f;\varkappa) = g(\varkappa).$$

Example 2.3.1: The mapping $A: C([0,1], K) \to C([0,1], K)$ defined as $A f(\varkappa) := f'(\varkappa)$, $f \in C[0,1]$ is an operator.

Definition 2.3.2: Assume X and Y are two function spaces and L is the operator defined in $L: X \to Y$. If f and g are any two functions taken from X and $\alpha_1, \alpha_2 \in \mathbb{R}$ be any numbers, L is called linear operator if it satisfy the condition;

$$L(\alpha_1 f + \alpha_2 g) = \alpha_1 L(f) + \alpha_2 L(g).$$

The X is called the domain of the operator L and shown as X=D(L) and Y is called the range of L and shown as R(L).

Example 2.3.2: The operator T defined by $T(\varkappa_1, \varkappa_2) = (\varkappa_2, \varkappa_1)$ is linear since;

$$\begin{aligned} T(\gamma\varkappa + \beta y) &= T\big(\gamma(\varkappa_1, \varkappa_2) + \delta(y_1, y_2)\big) \\ &= T(\gamma\varkappa_1 + \delta y_1, \gamma\varkappa_2 + \delta y_2) \\ &= T(\gamma\varkappa_2 + \delta y_2, \gamma\varkappa_1 + \delta y_1) \\ &= \gamma(\varkappa_2, \varkappa_1) + \delta(y_2, y_1) \\ &= \gamma T\varkappa + \delta T y \end{aligned}$$

is true $\forall\, \gamma, \delta \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$ where $x = (\varkappa_1, \varkappa_2)$ and $y = (y_1, y_2)$.

Definition 2.3.3: Let X^+ and Y^+ be the spaces of positive-valued functions taken from the space X and Y, respectively. That is

$$X^+ = \{f \in X : f(k) \geq 0, \forall\, k\}$$

$$Y^+ = \{g \in Y : g(k) \geq 0, \forall\, k\}.$$

If the operator L defined on X^+ maps each positive function into a function g in Y^+, that is,

$$L(f;\varkappa) \geq 0 \text{ when } f(k) \geq 0, \forall\, k \in D(f)$$

then the L is called a positive operator.

It means, if $L: X^+ \subset X \to Y^+ \subset Y$, then it is called linear positive operator.

Example 2.3.3: Szász operators

$$S_m(f;\varkappa) = e^{-m\varkappa} \sum_{j=0}^{\infty} f\left(\frac{j}{m}\right) \frac{(m\varkappa)^j}{j!}$$

defined on $C[0,1]$ are linear positive operators.

For all $\alpha_1, \alpha_2 \in \mathbb{R}$ and for all $f_1, f_2 \in C[0,1]$, we have;

$$S_m(\alpha_1 f_1 + \alpha_2 f_2; x) = e^{-m\varkappa} \sum_{j=0}^{\infty} (\alpha_1 f_1 + \alpha_2 f_2)\left(\frac{j}{m}\right) \frac{(m\varkappa)^j}{j!}$$

$$= e^{-m\varkappa}\, \alpha_1 \sum_{j=0}^{\infty} f_1\left(\frac{j}{m}\right) \frac{(m\varkappa)^j}{j!} + e^{-mx} \alpha_2 \sum_{k=0}^{\infty} f_2\left(\frac{j}{m}\right) \frac{(m\varkappa)^j}{j!}$$

$$= \alpha_1 S_m(f_1;\varkappa) + \alpha_2 S_m(f_2;\varkappa)$$

so S_m is linear.

For any $j \in \mathbb{N}$ and $x \in [0, A]$, $\frac{e^{-m\varkappa}(m\varkappa)^j}{j!} \geq 0$ and for $f \geq 0$, $S_m(f;\varkappa) \geq 0$. Therefore S_m is positive operator.

In the present part we shall mention two important properties of linear positive operators that are used in the proofs of some important theorems in the later chapters.

Property 1: Linear positive opeartors are monoton. It means, $f(\varkappa) \geq g(\varkappa)$ implies that $L(f;\varkappa) \geq L(g;\varkappa)$.

If, $\forall \varkappa \in \mathbb{R}$, $f(\varkappa) \geq g(\varkappa)$ then $f(\varkappa) - g(\varkappa) \geq 0$.

L is a positive operator so we can say,

$$L(f - g;\varkappa) \geq 0$$

also L is a linear, so

$$L(f;\varkappa) - L(g;\varkappa) \geq 0,$$

and

$$L(f;\varkappa) \geq L(g;\varkappa).$$

Property 2: Let L be a linear positive operator. Then it satisfies the inequality;

$$|L(f;\varkappa)| \leq L(|f|;\varkappa).$$

$\forall$ k $\in [a, b]$, $\quad - |f(k)| \leq f(k) \leq |f(k)|$.

By using monotonicity of the linear positive operators, we can write;

$$-L(|f|;\varkappa) \leq L(f;\varkappa) \leq L(|f|;\varkappa).$$

Therefore we get;

$$|L(f;\varkappa)| \leq L(|f|;\varkappa).$$

Definition 2.3.4: If f has derivatives of all orders at $\varkappa = x_0$, then the Taylor series for the function f at $\varkappa_0$ is,

$$\sum_{j=0}^{\infty} \frac{f^j(\varkappa_0)}{j!}(\varkappa - \varkappa_0)^j.$$

When $\varkappa = 0$ we get;

$$\sum_{j=0}^{\infty} \frac{f^j(0)}{j!}\varkappa^j$$

and this series is well-known Maclaurin series.

Definition 2.3.5: Assume that $\mathrm{L}: \mathrm{C}[0,1] \rightarrow \mathrm{C}[0,1]$. Hölder's inequality for $\rho, \mathrm{q} > 1, \frac{1}{\rho}+\frac{1}{\mathrm{q}}=1$ and $\mathrm{f}, g \in \mathrm{C}[0,1], \varkappa \in [0,1]$ is provided as follows;

$$L(|fg|; \varkappa) \leq L(|\mathrm{f}|^{\rho}; \varkappa)^{\frac{1}{\rho}} L(|\mathrm{f}|^{q}; \varkappa)^{\frac{1}{q}}.$$

This is known Cauchy Schwarz when ρ =q=2.

2.4 Rate Of Convergence

Theory of rate of convergence explores whether a function in a given space can be approximated with a family of functions belonging to the same space with good properties. There exists an iterative algorithm that is trying to find the maximum or minimum. In this section we will study how long it will take to reach that optimal value by using rate of converges.

We can find rate of converge to series α_{m} $(\mathrm{m} \rightarrow \infty)$ which satisfies inequality;

$$|\mathrm{L}_{\mathrm{m}}(\mathrm{f}; \varkappa) - \mathrm{f}(\varkappa)| \leq \mathrm{c}\alpha_{n}, \qquad c \in \mathbb{R}^{+}.$$

Definition 2.4.1: Let $\mathrm{f} \in \mathrm{C}[\mathrm{a}, \mathrm{b}]$, for $\zeta > 0$,

$$\omega(\mathrm{f}; \zeta) = \omega(\zeta) = \max_{\substack{\varkappa, \mathrm{k} \in [\mathrm{a}, \mathrm{b}] \\ |\mathrm{k}-\varkappa| \leq \zeta}} |\mathrm{f}(\mathrm{k}) - \mathrm{f}(\varkappa)|$$

The function $\omega(\zeta)$ defined in this way is called the modulus of continuity of f.

Theorem 2.4.1: The modulus of continuity $\omega(\mathrm{f}; \zeta)$ satisfies following properties:

I. $\omega(\mathrm{f}; \zeta) \geq 0$.

II. If $\zeta_1 \leq \zeta_2$ then $\omega(\mathrm{f}; \zeta_1) \leq \omega(\mathrm{f}; \zeta_2)$.

III. $\lim_{\zeta \rightarrow 0} \omega(\mathrm{f}; \zeta) = 0$.

IV. For $m \in \mathbb{N}$, $\omega(f; m\zeta) \le m.\omega(f;\zeta)$.

V. For $\lambda \in \mathbb{R}^+$, $\omega(f;\lambda\zeta) \le (\lambda+1)\omega(f;\zeta)$.

VI. $|f(k) - f(\varkappa)| \le \omega(f; |k-\varkappa|)$.

VII. $|f(k) - f(\varkappa)| \le \left(\frac{|k-\varkappa|}{\zeta} + 1\right)\omega(f;\zeta)$.

Proof:

I. It is clear that it is maximum of absolute value, by the definition of modulus of continuity.

II. For $\zeta_1 \le \zeta_2$, the region of $|k-\varkappa| \le \zeta_2$ is greater than the region $|k-\varkappa| \le \zeta_1$. Proof is clear, such that when region grow then the supremum grow.

III. f is continuous, $\forall\, \varepsilon \ge 0$, when $|\varkappa_1 - \varkappa_2| < \eta$, there exists at least one $\eta > 0$ such that;

$$|f(\varkappa_1) - f(\varkappa_2)| < \varepsilon.$$

So,

$$\sup|f(\varkappa_1) - f(\varkappa_2)| < \varepsilon.$$

By the definition of $\omega(f;\eta)$;

$$\omega(f;\eta) < \varepsilon$$

Use (II) for $\zeta < \eta$, we can say that;

$$\omega(f;\zeta) < \varepsilon.$$

IV. By the definition of the modulus of continuity for $m \in \mathbb{N}$. We can say that;

$$\omega(f; m\zeta) = \sup_{\substack{\varkappa, k \in [a,b] \\ |k-\varkappa| \le m\zeta}} |f(k) - f(\varkappa)|.$$

If

$$|k - \varkappa| \le m\zeta,$$

then

$$\varkappa - m\zeta \le k \le \varkappa + m\zeta.$$

By choice $k = \varkappa + mh$, for $|h| \leq \zeta$ we can say that;

$$\omega(f; m\zeta) = \sup_{\substack{\varkappa, k \in [a,b] \\ |h| \leq \zeta}} |f(\varkappa + mh) - f(\varkappa)|.$$

On the other hand;

$$\sup_{\substack{\varkappa, k \in [a,b] \\ |h| \leq \zeta}} |f(\varkappa + mh) - f(x)| = \sup_{\substack{\varkappa, k \in [a,b] \\ |h| \leq \zeta}} \left| \sum_{j=0}^{m} [f(\varkappa + (j+1)h) - f(\varkappa + jh)] \right|.$$

Apply the triangle inequality to the right of equation, we get;

$$\sup_{\substack{\varkappa, k \in [a,b] \\ |h| \leq \zeta}} |f(\varkappa + mh) - f(x)| \leq \sum_{j=0}^{m-1} \sup_{\substack{\varkappa, k \in [a,b] \\ |h| \leq \zeta}} |f(\varkappa + (j+1)h) - f(\varkappa + jh)|$$

$$\leq \omega(f; \zeta) + \omega(f; \zeta) + \cdots + \omega(f; \zeta)$$

$$= m\omega(f; \zeta).$$

V. If the whole part of the number $\lambda \in \mathbb{R}^+$ is denoted by $[|\lambda|]$, the inequality $[|\lambda|] < \lambda < [|\lambda|] + 1$ is valid by the definition of the integer function. Since $[|\lambda|]$ is the positive integer, if the property (IV) is applied to the right side of inequality,

$$\omega(f; [|\lambda|] + 1)\zeta \leq ([|\lambda|] + 1)\omega(f; \zeta)$$

is obtained.

On the other hand, since $[|\lambda|] + 1 < \lambda + 1$ for $\forall \lambda \in \mathbb{R}^+$ then;

$$\omega(f; [|\lambda|] + 1)\zeta \leq (\lambda + 1)w(f; \zeta).$$

Using (1.4.1) from here, we can write

$$\omega(f; \lambda\zeta) \leq (\lambda + 1)\omega(f; \zeta)$$

VI. If we select $\zeta = |k - \varkappa|$ in expression $\omega(f; \zeta)$, we get;

$$\omega(f; |k - \varkappa|) = \sup_{\varkappa \in [a,b]} |f(k) - f(\varkappa)|.$$

So proof is clear that supremum of $|f(k) - f(\varkappa)|$ is $\omega(f; |k - \varkappa|)$.

VII. From (VI), we can write;

$$|f(k) - f(\varkappa)| \leq \omega\left(f; \frac{|k - \varkappa|}{\zeta} \zeta\right).$$

Using the property (V) of this inequality the proof is clear as follows;

$$|f(t) - f(\varkappa)| \leq \left(\frac{|k - \varkappa|}{\zeta} + 1\right) \omega(f; \zeta).$$

Definition 2.4.2: For $0 < \alpha \leq 1$ and $N > 0$;

$$|f(k) - f(\varkappa)| \leq N|k - \varkappa|^{\alpha}.$$

The functions satisfying the above condition are called Lipschitz class functions. The set of all Lipschitz class functions are shown as $\text{lip}_N(\alpha)$. Moreover, N is called Lipschitz constant and f is Lipschitz class function i.e. $f \in \text{lip}_N(\alpha)$.

Example 2.4.1: If $\omega(f, \zeta) \leq N\zeta$ for some constant N >0, then Lipschitz condition is satisfied for f; the least value N satisfying such inequality is the Lipschitz constant of f.

Example 2.4.2: If $\omega(f, \zeta) \leq N\zeta^{\alpha}$ for some constants N >0, $\alpha \in [0,1]$ then f satisfies the Hölder condition of factor α.

Chapter 3

KOROVKIN THEOREM AND BERNSTEIN OPERTATORS

The subject of linear positive operators and approximation in contemporary functional analysis and theory of functions is a research area that has emerged in the last sixty years. At present, approaches to the theory of approximations are mostly based on real valued continuous functions with the help of algebraic polynomials. Bohman stated and proved that linear positive operators only need to fulfill three conditions in order to converge properly to a continuous function in the closed interval [0,1], (Bohman 1952). Later in 1953, Korovkin proved the same theorem by expanding the range for operators of integral type, (Korovkin 1953). Therefore this theorem is more commonly known as the Bohman-Korovkin theorem. After that, Korovkin gave a very important theorem for the approximation of linear positive operators, since Bernstein operators are also linear positive operators, studies on this have gained momentum. The Bernstein operator $B_m(f;\varkappa)$ is an operartor of $\varkappa$, which order is m^{th}. It was constructed by Bernstein to give a simple proof of Weiestrass's approximation theorem.

3.1 Korovkin Theorem

In 1952, H.Bohman studied the problem of converging linear positive operators in the form of sum to the continuous function f in [0,1]. However, the value of the operators investigated by H.Bohman is independent of the values outside the [0,1] range of the function f. Therefore, in 1953, P.P.Korovkin proved a general theorem and showed that the conditioning expressed by Bohman is also valid in the general case.

Definition 3.1.1: Linear Positive Operator Series $L_m(f;\varkappa)$ are uniformly converges to f if and only if;

$$L_m(1;\varkappa) \rightrightarrows 1, \quad (3.1.1)$$

$$L_m(k;\varkappa) \rightrightarrows \varkappa, \quad (3.1.2)$$

$$L_m(k^2;\varkappa) \rightrightarrows \varkappa^2. \quad (3.1.3)$$

Theorem 3.1.1: If a linear positive operator $L_m(f;\varkappa)$ satisfies Bohman's conditions on closed interval [a,b] when $m \to \infty$ then for any continuous function f on [a,b] then

$$L_m(f;\varkappa) \rightrightarrows f(\varkappa).$$

Proof:

Let's $f \in C[a,b]$. Then $\forall\, \varepsilon > 0,\ \exists\, \zeta > 0$ such that $\forall\, k \in \mathbb{R}$ and $\forall\, \varkappa \in [a,b]$

$$|f(k) - f(\varkappa)| < \varepsilon, \quad (3.1.4)$$

provided that $|k - \varkappa| < \zeta$.

Moreover, $\forall\, \varkappa \in \mathbb{R},\ \exists\, N > 0$ such that;

$$|f(\varkappa)| \leq N.$$

Since f is bounded and apply the triangle inequality, if $|k - \varkappa| \geq \zeta$ we get;

$$|f(k) - f(\varkappa)| \leq |f(k)| + |f(\varkappa)| \leq N + N \leq 2N.$$

and

$$|k - \varkappa| \geq \zeta \quad \Rightarrow \quad (k - \varkappa)^2 \geq \zeta^2.$$

So,

$$\frac{(k-\varkappa)^2}{\zeta^2} \geq 1 \quad \Rightarrow \quad \frac{2N}{\zeta^2}(k - \varkappa)^2 \geq 2N. \quad (3.1.5)$$

By using inequality (3.2.4) & (3.2.5) we get;

$$|f(k) - f(\varkappa)| < \varepsilon + 2N < \varepsilon + \frac{2N}{\zeta^2}(k - \varkappa)^2. \quad (3.1.6)$$

Since $L_m: C[a,b] \to C[a,b]$ is linear positive, it follows that;

$$\|L_m(f;\varkappa)-f(\varkappa)\|_{C[a,b]} = \|L_m(f(k)-f(\varkappa);\varkappa)+f(\varkappa)(L_m(1;\varkappa)-1)\|_{C[a,b]}$$

$$\leq \|L_m(f(k)-f(\varkappa);\varkappa)\|_{C[a,b]} + \|f\|_{C[a,b]}\|L_m(1;\varkappa)-1\|_{C[a,b]}$$

$$\leq \|L_m(|f(k)-f(\varkappa)|;\varkappa\|_{C[a,b]} + \|f\|_{C[a,b]}\|L_m(1;\varkappa)-1\|_{C[a,b]}.$$

Then from (3.1.1), when $m \to \infty, \exists\, \varepsilon_m$ such that $\varepsilon_m \to 0$. So;

$$\|f\|_{C[a,b]}\|L_m(1;\varkappa)-1\|_{C[a,b]} \leq \varepsilon_m.$$

In this case;

$$\|L_m(f;\varkappa)-f(\varkappa)\|_{C[a,b]} \leq \|L_m(|f(k)-f(\varkappa)|;\varkappa\|_{C[a,b]} + \varepsilon_m \tag{3.1.7}$$

can be written.

Let us show that;

$$\lim_{m\to\infty}\|L_m(|f(k)-f(\varkappa)|;\varkappa)\|_{C[a,b]} = 0.$$

By using inequality (3.1.6) and (3.1.7) we have,

$$|f(k)-f(\varkappa)| \leq \varepsilon + \frac{2N}{\zeta^2}(k-\varkappa)^2.$$

Taking L_m for both sides,

$$L_m(|f(k)-f(\varkappa)|;\varkappa) \leq L_m(\varepsilon;\varkappa) + L_m\left(\frac{2N}{\zeta^2}(k-\varkappa)^2;\varkappa\right)$$

$$= \varepsilon L_m(1;\varkappa) + \frac{2N}{\zeta^2}L_m((k-\varkappa)^2;\varkappa)$$

$$= \varepsilon L_m(1;\varkappa) + \frac{2N}{\zeta^2}L_m(k^2-2k\varkappa+\varkappa^2;\varkappa)$$

$$= \varepsilon L_m(1;\varkappa) + \frac{2N}{\zeta^2}L_m(k^2;\varkappa) - \frac{4N}{\zeta}\varkappa L_m(k;\varkappa) + \frac{2N}{\zeta^2}\varkappa^2 L_m(1;\varkappa)$$

$$= \varepsilon[L_m(1;\varkappa)-1+1] + \frac{2N}{\zeta^2}[L_m(k^2;\varkappa)-\varkappa^2+\varkappa^2] - \frac{4N}{\zeta^2}\varkappa[L_m(k;\varkappa)-\varkappa+\varkappa]$$

$$+\frac{2N}{\zeta^2}\varkappa^2[L_m(1;\varkappa)-1+1]$$

$$= \varepsilon[L_m(1;\varkappa)-1] + \varepsilon_m + \frac{2N}{\zeta^2}[L_m(k^2;\varkappa)-\varkappa^2] + \frac{2N}{\zeta^2}\varkappa^2 - \frac{4N}{\zeta^2}\varkappa[L_m(k;\varkappa)-\varkappa]$$

$$-\frac{4N}{\zeta^2}\varkappa^2 + \frac{2N}{\zeta^2}\varkappa^2[L_m(1;\varkappa)-1] + \frac{2N}{\zeta^2}\varkappa^2$$

$$= \varepsilon + \left(\varepsilon_m + \frac{2N}{\zeta^2}\varkappa^2\right)[L_m(1;\varkappa)-1] + \frac{2N}{\zeta^2}[L_m(k^2;\varkappa)-\varkappa^2] - \frac{4N}{\zeta^2}\varkappa[L_m(k;\varkappa)-\varkappa].$$

Let's say;

$$\varepsilon + \frac{2N}{\zeta^2}\varkappa^2 = g(\varkappa) \quad \text{and} \quad -\frac{4N}{\zeta^2}\varkappa = h(\varkappa) \quad \text{where;}$$

$$\forall \varkappa \in [a,b], \qquad g(\varkappa) \leq \sup_{\varkappa \in [a,b]} |g(\varkappa)| = b$$

and

$$h(\varkappa) \leq \sup_{\varkappa \in [a,b]} |h(\varkappa)| = c\,,$$

then;

$$L_m(|f(k) - f(\varkappa)|; \varkappa) \leq \varepsilon + b[L_m(1;\varkappa) - 1] + \frac{2N}{\zeta^2}\left[L_m\left(k^2;\varkappa\right) - \varkappa^2\right] + c[L_m(k;\varkappa) - \varkappa].$$

Taking maximum norm we get,

$$\|L_m(|f(k) - f(\varkappa)|;\varkappa)\|_{C[a,b]} \leq \varepsilon_m + b\|L_n(1;\varkappa) - 1\|_{C[a,b]} + \frac{2N}{\zeta^2}\|L_m(k^2;\varkappa) -$$

$$-\varkappa^2\|_{C[a,b]} + c\|L_m(k;\varkappa) - \varkappa\|_{C[a,b]}.$$

Then by using Bohman's conditions;

$$\|L_m(|f(k) - f(\varkappa)|;\varkappa)\|_{C[a,b]} \leq \varepsilon_n$$

$$\lim_{m\to\infty}\|L_m(|f(k) - f(\varkappa)|;\varkappa)\|_{C[a,b]} = 0, \text{ since } \lim_{m\to\infty}\varepsilon_m = 0.$$

Also it satisfies inequality (2.2.7)

$$\left||L_m(f;\varkappa) - f(\varkappa)|\right|_{C[a,b]} \leq \left||L_m(|f(k) - f(\varkappa)|;\varkappa|\right|_{C[a,b]} + \varepsilon_m$$

$$\left||L_m(f;\varkappa) - f(\varkappa)|\right|_{C[a,b]} = 0.$$

So the Korovkin Theorem is provided.

3.2 Bernstein Operators

The main result in the development of the theory of approximation, founded in 1885 by the German Mathematician K. Weierstrass, for each $f \in C[0,1]$, $\exists$ a polynomial $P(\varkappa)$ such that for any $\varepsilon > 0$, the assertion that,

$$|f(\varkappa) - P(\varkappa)| < \varepsilon$$

$x \in [a, b]$ is true. This theorem is related to the fact that the space of polynomials is dense in C[a,b]. Weierstrass's first proof was quite difficult to understand because it was complex and long. Such complexity has motivated many mathematicians to find easy, more simpler and understandable proofs.

The Bernstein operators are given as follows;

$$B_m(f; \varkappa) = \sum_{j=0}^{m} f\left(\frac{j}{m}\right) b_{m,j}(\varkappa).$$

For any function f defined in [0,1] were introduced in 1912 to give a simpler proof of Weierstrass's approximation theory. It was created by the Russian mathematician S.N. Bernstein. The method of defining Bernstein polynomials helped to define many new sets of polynomials that are approximating to continuous functions.

Definition 3.2.1: Assume $f \in C[0,1]$ is given. For $0 \leq \varkappa \leq 1$, m^{th} order Bernstein Operators are defined as;

$$B_m(f; \varkappa) = \sum_{j=0}^{m} f\left(\frac{j}{m}\right) b_{m,j}(\varkappa),$$

where

$$b_{m,j}(\varkappa) = C_m^j \varkappa^j (1 - \varkappa)^{m-j},$$

and

$$C_m^j = \binom{m}{j} = \frac{m!}{(m-j)!j!}.$$

The basic structure of these polynomials depends on the binomial formula;

$$(a + b)^m = \sum_{j=0}^{m} C_m^j a^j b^{m-j}$$

where $a, b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. If $a = \varkappa$ and $b = 1 - \varkappa$ are taken with $\varkappa \in [0,1]$ in this formula, we get;

$$(\varkappa + 1 - \varkappa)^m = (1)^m = 1 = \sum_{j=0}^{m} C_m^j \varkappa^j (1 - \varkappa)^{m-j}.$$

Lemma 3.2.1: Bernstein operators are linear and positive.

Proof:

$\forall \alpha_1, \alpha_2 \in \mathbb{R}$ and $\forall f_1, f_2 \in C[0,1]$ there exists;

$$B_m(\alpha_1 f_1 + \alpha_2 f_2; \varkappa) = \sum_{j=0}^{m} (\alpha_1 f_1 + \alpha_2 f_2)\left(\frac{j}{m}\right) C_m^j \varkappa^j (1 - \varkappa)^{m-j}$$

$$= a_1 \sum_{j=0}^{m} f_1\left(\frac{j}{m}\right) C_m^j \varkappa^j (1 - \varkappa)^{m-j} + a_2 \sum_{j=0}^{m} f_2\left(\frac{j}{m}\right) C_m^j \varkappa^j (1 - \varkappa)^{m-j}$$

$$= a_1 B_m(f_1; \varkappa) + a_2 B_m(f_2; \varkappa).$$

So B_m is linear.

For $\varkappa \in [0,1], \varkappa^j (1 - \varkappa)^{m-j} \geq 0$ and for $f \geq 0, B_m(f; \varkappa) \geq 0$. Therefore B_m is positive.

Theorem 3.2.1: Let's f be continuous then Bernstein operators are uniformly converges on [0,1] to $f(x)$.

In other words if $f \in C[0,1]$, then;

$$B_m(f; \varkappa) \rightrightarrows f(\varkappa), \qquad \varkappa \in [0,1].$$

Proof: Now, let us investigate the conditions of the Korovkin's;

$$B_m(1; \varkappa) = \sum_{j=0}^{m} \binom{m}{j} \varkappa^j (1 - \varkappa)^{m-j}$$

$$= (1 - \varkappa + \varkappa)^m$$

$$= 1$$

$$B_m(k;\varkappa) = \sum_{j=0}^{m} \frac{j}{m}\binom{m}{j}\varkappa^j(1-\varkappa)^{m-j}$$

$$= \varkappa\sum_{j=1}^{m}\binom{m-1}{j-1}\varkappa^{j-1}(1-\varkappa)^{m-j}$$

$$= \varkappa\sum_{j=1}^{m-1}\binom{m-1}{j}\varkappa^{j}(1-\varkappa)^{m-1-j}$$

$$= \varkappa(1-\varkappa+\varkappa)^{m-1}$$

$$= \varkappa.$$

$$B_m(k^2;\varkappa) = \sum_{k=0}^{m}\frac{j^2}{m^2}\binom{m}{j}\varkappa^j(1-\varkappa)^{m-j}$$

$$= \varkappa\sum_{j=1}^{m}\frac{j}{m}\binom{m-1}{j-1}\varkappa^{j-1}(1-\varkappa)^{m-j}$$

$$= \varkappa\sum_{k=1}^{m}\frac{j-1}{m}\binom{m-1}{j-1}\varkappa^{j-1}(1-\varkappa)^{m-j} + \frac{\varkappa}{m}\sum_{j=1}^{m}\binom{m-1}{j-1}\varkappa^{j-1}(1-\varkappa)^{m-j}$$

$$= \varkappa^2\frac{(m-1)}{m}\sum_{j=2}^{m}\binom{m-2}{j-2}\varkappa^{j-2}(1-\varkappa)^{m-j} + \frac{\varkappa}{m}\sum_{j=0}^{m-1}\binom{m-1}{j}\varkappa^{j}(1-\varkappa)^{m-1-j}$$

$$= \varkappa^2\frac{(m-1)}{m}\sum_{j=0}^{m-2}\binom{m-2}{j}\varkappa^{j}(1-\varkappa)^{m-2-j} + \frac{\varkappa}{m}(1-\varkappa+\varkappa)^{m-1}$$

$$= \varkappa^2\frac{(m-1)}{m}(1-\varkappa+\varkappa)^{m-2} + \frac{\varkappa}{m}$$

$$= \varkappa^2\frac{(m-1)}{m} + \frac{\varkappa}{m}$$

$$= \varkappa^2 + \frac{\varkappa-\varkappa^2}{m}\ .$$

Therefore;

$$B_m(1;\varkappa) = 1$$

$$B_m(k;\varkappa) = \varkappa$$

$$B_m(k^2;\varkappa) = \varkappa^2 + \frac{\varkappa - \varkappa^2}{m}$$

Then, it shows that;

1- $\lim_{m\to\infty} \|B_m(1;\varkappa) - 1\|_{C[0,1]} = 0$

2- $\lim_{m\to\infty} \|B_m(k;\varkappa) - \varkappa\|_{C[0,1]} = 0$

3- $\lim_{m\to\infty} \|B_m(k^2;\varkappa) - \varkappa^2\|_{C[0,1]} = \lim_{m\to\infty} \left\|\varkappa^2 + \frac{\varkappa-\varkappa^2}{m} - \varkappa^2\right\|_{C[0,1]}$

$$= \lim_{m\to\infty} \max_{0\le\varkappa\le 1} \left\|\frac{\varkappa - \varkappa^2}{m}\right\| = \lim_{m\to\infty} \frac{1}{4m} = 0$$

For $f \in C[0,1]$,

$$\|B_m(f;\varkappa) - f(\varkappa)\|_{C[0,1]} \to 0,$$

by the Korovkin Theorem.

Theorem 3.2.2: $B_m(f;\varkappa)$ defined in the theorem (3.1.1), the inequality

$$|B_m(f;\varkappa) - f(\varkappa)| \le \frac{3}{2}\omega\left(f; m^{-\frac{1}{2}}\right)$$

is provided for the continuous function f on the [0,1].

Proof: Let $b_{m,j}(\varkappa) = \binom{m}{j}\varkappa^j(1-\varkappa)^{m-j}$, from the definition of Bernstein operator and linearity property;

$$|B_m(f;\varkappa) - f(\varkappa)| = \left|\sum_{j=0}^{m} f\left(\frac{j}{m}\right) b_{m,k}(\varkappa) - f(\varkappa)\right|$$

$$= \left|\sum_{j=0}^{m} \left(f\left(\frac{j}{m}\right) - f(\varkappa)\right) b_{m,j}(\varkappa)\right|$$

$$\leq \sum_{j=0}^{m} \left| f\left(\frac{j}{m}\right) - f(\varkappa) \right| b_{m,j}(\varkappa)$$

(3.2.8)

is obtained. By choosing $k = \frac{j}{m}$ in the property (VII) of the modulus of continuity;

$$\left| f\left(\frac{j}{m}\right) - f(\varkappa) \right| \leq \left(\frac{\left|\frac{j}{m} - \varkappa\right|}{\zeta_m} + 1 \right) \omega(f; \zeta_m)$$

can be written. If we write this instead in (3.2.8) we get;

$$|B_m(f;\varkappa) - f(\varkappa)| \leq \sum_{j=0}^{m} \left(\frac{\left|\frac{j}{m} - \varkappa\right|}{\zeta_m} + 1 \right) \omega(f; \zeta_m) b_{m,j}(\varkappa)$$

$$= \omega(f; \zeta_m) \left[b_{m,j}(\varkappa) + \frac{1}{\zeta_m} \sum_{j=0}^{m} \left| \frac{j}{m} - \varkappa \right| b_{m,j}(\varkappa) \right]$$

Apply the Cauchy Schwarz to the second part of the sum, we get;

$$|B_m(f;\varkappa) - f(\varkappa)| \leq \omega(f; \zeta_m) \left[1 + \frac{1}{\zeta_m} \left(b_{m,j}(\varkappa) \right)^{\frac{1}{2}} \left(\sum_{j=0}^{m} \left(\frac{j}{m} - \varkappa \right)^2 b_{m,j}(\varkappa) \right)^{\frac{1}{2}} \right]$$

In this case;

$$|B_m(f;\varkappa) - f(\varkappa)| \leq \omega(f; \zeta_m) \left[1 + \frac{1}{\zeta_m} \left(\sum_{j=0}^{m} \left(\frac{j}{m} - \varkappa \right)^2 b_{m,j}(\varkappa) \right)^{\frac{1}{2}} \right].$$

(3.2.9)

Since $\sum_{j=0}^{m} \left(\frac{j}{m} - \varkappa \right)^2 b_{m,j}(\varkappa) = B_m((k-\varkappa)^2; \varkappa)$ and B_m is linear, then ;

$$B_m((k-\varkappa)^2;\varkappa) = B_m(k^2;\varkappa) - 2x B_m(k;\varkappa) + x^2 B_m(1;\varkappa)$$

$$= \varkappa^2 + \frac{\varkappa(1-\varkappa)}{m} - 2\varkappa^2 + \varkappa^2$$

$$= \frac{\varkappa(1-\varkappa)}{\mathrm{m}}$$

If we write this instead in (3.2.9) we get;

$$|\mathrm{B_m}(\mathrm{f};\varkappa) - \mathrm{f}(\varkappa)| \leq \omega(\mathrm{f};\zeta_{\mathrm{m}}) \left[1 + \frac{1}{\zeta_{\mathrm{m}}} \frac{\sqrt{\varkappa(1-\varkappa)}}{\sqrt{\mathrm{m}}}\right]$$

$\forall\, \varkappa \in [0,1]$ it becomes;

$$\sqrt{\varkappa(1-\varkappa)} \leq \max_{0\leq\varkappa\leq1} \sqrt{\varkappa(1-\varkappa)} = \tfrac{1}{2}.$$

From here,

$$|\mathrm{B_m}(\mathrm{f};\varkappa) - \mathrm{f}(\varkappa)| \leq \omega(\mathrm{f};\zeta_n) \left[1 + \frac{1}{\zeta_{\mathrm{m}}} \frac{1}{2\mathrm{m}^{\frac{1}{2}}}\right]$$

is obtained.

If $\zeta_{\mathrm{m}} = \mathrm{m}^{-\frac{1}{2}}$ is chosen, we get the inequality;

$$|\mathrm{B_m}(\mathrm{f};\varkappa) - \mathrm{f}(\varkappa)| \leq \omega\left(\mathrm{f};\mathrm{m}^{-\frac{1}{2}}\right)\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{2}\omega\left(\mathrm{f};\mathrm{m}^{-\frac{1}{2}}\right).$$

This inequality shows us that the rate of converges of the operator $\mathrm{B_{m,j}}$ to the function f, with $\varkappa \in [a,b]$ is smaller than the rate of converges of $\mathrm{m}^{-\frac{1}{2}}$ to 0.

Chapter 4

BLENDING TYPE BERNSTEIN OPERATORS

Blending type operators are the presence of more than one operator in one place. In this chapter, we examine the new generalized Bernstein operators with parameter α and their convexity, monotonicity and other important properties.

4.1 The New Generalized Bernstein Operators

Bernstein polynomials and their modifications have been intensely studied because of their useful structure. Chen et al. introduced a generalization of the Bernstein operators. They proved the rate of convergence and shape preserving properties for these operators.

Definition 4.1.1: Assume that f is a continuous function on $[0,1]$. $\forall$ $m \in \mathbb{Z}^+$ and any real constant $\alpha \in [0,1]$, the $\alpha -$ Bernstein operators are defined for f as:

$$T_{m,\alpha}(f;\varkappa) = \sum_{j=0}^{m} f_j P_{m,j}^{(\alpha)}(\varkappa) \qquad (4.1.1)$$

where $f_j = f\left(\frac{j}{m}\right)$. For $j \in [0,1]$, $P_{m,j}^{(\alpha)}(\varkappa)$ is defined by $P_{1,0}^{(\alpha)}(\varkappa) = 1 - \varkappa$, $P_{1,0}^{(\alpha)}(\varkappa) = \varkappa$ and

$$P_{m,j}^{(\alpha)}(\varkappa) = \left[\binom{m-2}{j}(1-\alpha)\varkappa + \binom{m-2}{j-2}(1-\alpha)(1-\varkappa) + \binom{m}{j}\alpha\varkappa(1-\varkappa)\right]\varkappa^{j-1}(1-\varkappa)^{m-j-1}$$

The $\alpha -$ Bernstein operators include classical Bernstein operator when $\alpha = 1$ where;

$$P_{m,j}^{(1)} = \binom{m}{j}\varkappa^j(1-\varkappa)^{m-j}.$$

Therefore α -Bernstein operators have following property;

$$T_{m,1}(f;\varkappa) = \sum_{j=0}^{m} f_j \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j} = B_m(f;\varkappa).$$

Lemma 4.1.1: For all $m \in \mathbb{N}, m \geq 1$ and independent of α, $T_{m,\alpha}(f;\varkappa)$ satisfies following properties;

I. f is interpolated by the $\alpha-$Bernstein operator at both endpoints [a,b];

$$T_{m,\alpha}(f;0) = f(0) \text{ and } T_{m,\alpha}(f;1) = f(1).$$

II. The $\alpha -$ Bernstein operator is linear operator, such that;

$$T_{m,\alpha}(\beta f + \gamma g) = \beta T_{m,\alpha}(f) + \gamma T_{m,\alpha}(g) \tag{4.1.2}$$

$\forall\, f, g \in C[0,1]$ where defined on [0,1] and all $\beta, \gamma \in \mathbb{R}$.

We can express (4.1.1) in the following form to discuss other properties of these operators;

$$T_{m,\alpha}(f;\varkappa) = (1-\alpha)(k_1 + k_2) + \alpha \sum_{j=0}^{m} f_j \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j}$$

where

$$k_1 = \sum_{j=1}^{m} f_j \binom{m-2}{j} \varkappa^j (1-\varkappa)^{m-j-1},$$

$$k_2 = \sum_{j=1}^{m} f_j \binom{m-2}{j-2} x^{j-1} (1-\varkappa)^{m-j}.$$

When j =m in k_1 and j =0 in k_2, they are both zero. They can be expressed as;

$$k_1 = \sum_{j=1}^{m-1} f_j \binom{m-2}{j} \varkappa^j (1-\varkappa)^{m-j-1},$$

$$k_2 = \sum_{j=1}^{m} f_j \binom{m-2}{j-2} \varkappa^{j-1}(1-\varkappa)^{m-j}$$

where $\binom{m-2}{m-1} = 0$ in k_1 and $\binom{m-2}{-1} = 0$ in k_2. Replace j by j +1 in the k_2, we get;

$$k_2 = \sum_{j=1}^{m-1} f_{j+1} \binom{m-2}{j-2} \varkappa^{j}(1-\varkappa)^{m-j-1}.$$

Therefore we obtain;

$$k_1 + k_2 = \sum_{j=1}^{m-1} \left[f_j \binom{m-2}{j} + f_{j+1} \binom{m-2}{j-2} \right] \varkappa^{j}(1-\varkappa)^{m-j-1}.$$

Seeing that;

$$\binom{m-2}{j} = \left(1 - \frac{j}{m-1}\right)\binom{m-1}{j} \text{ and } \binom{m-2}{j-1} = \frac{j}{m-1}\binom{m-1}{j}$$

We have

$$k_1 + k_2 = \sum_{j=1}^{m-1} \left[\lambda_j \binom{m-1}{j} \right] \varkappa^{j}(1-\varkappa)^{m-j-1},$$

(4.1.3)

Where λ_j is a linear combination of f_j and f_{j+1} such as;

$$\lambda_j = \left(1 - \frac{j}{m-1}\right) f_j + \frac{j}{m-1} f_{j+1}. \qquad (4.1.4)$$

Theorem 4.1.1: We can express $T_{m,\alpha}(f;\varkappa)$ as:

$$T_{m,\alpha}(f;\varkappa) = (1-\alpha) \sum_{j=0}^{m-1} \lambda_j \binom{m-1}{j} \varkappa^{j}(1-\varkappa)^{m-j-1} + \alpha \sum_{j=0}^{m} f_j \binom{m}{j} \varkappa^{j}(1-\varkappa)^{m-j}$$

(4.1.5)

where

$$\lambda_j = \left(1 - \frac{j}{m-1}\right) f_j + \frac{j}{m-1} f_{j+1}.$$

Lemma 4.1.2: The following identities are valid for these operators.

i. $T_{m,\alpha}(1; \varkappa) = 1.$ (4.1.6)

ii. $T_{m,\alpha}(k; \varkappa) = \varkappa.$ (4.1.7)

Proof:

i. When we use (4.1.4) and (4.1.5), if $f(\varkappa) \equiv 1$ then $f_j = \lambda_j = 1$ and

$$T_{m,\alpha}(f; \varkappa) = (1-\alpha) \sum_{j=0}^{m-1} \binom{m-1}{j} \varkappa^j (1-\varkappa)^{n-j-1} + \alpha \sum_{j=0}^{m} \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j}$$

Therefore, for the constant function 1 is, this operator;

$$T_{m,\alpha}(1; \varkappa) = (1-\alpha) \sum_{j=0}^{m-1} P_{m-1,j}(\varkappa) + \alpha \sum_{j=0}^{m} P_{m,j}(\varkappa) = 1$$

ii. If $f(\varkappa)=\varkappa$, then $f_j = \frac{j}{m}$. So;

$$\lambda_j = \left(1 - \frac{j}{m-1}\right) \frac{j}{m} + \frac{j}{m-1} \frac{j+1}{m} = \frac{j}{m-1}$$

and

$$T_{m,\alpha}(\tau; \varkappa) = (1-\alpha) \sum_{j=0}^{m-1} \frac{j}{m-1} \binom{m-1}{j} \varkappa^j (1-\varkappa)^{m-j-1}$$

$$+ \alpha \sum_{j=0}^{m} \frac{j}{m} \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j}$$

$$= (1-\alpha) \sum_{j=0}^{m-1} \frac{j}{m-1} P_{m-1,j}(\varkappa) + \alpha \sum_{j=0}^{m} \frac{j}{m} P_{m,j}(\varkappa)$$

$$= \varkappa.$$

These complete the proof.

Lemma 4.1.3: The α-Bernstein operators reproduce linear polynomials $\forall\, \beta, \gamma \in \mathbb{R}$ such that;

$$T_{m,\alpha}(\beta\varkappa + \gamma; \varkappa) = \beta\varkappa + \gamma.$$

Proof: It can be proved easily by using (4.1.6), (4.1.7) and its linearity. Namely;

$$T_{m,\alpha}(\beta\varkappa + \gamma; \varkappa) = \beta\varkappa + \gamma.$$

Lemma 4.1.4: The α-Bernstein operator is a positive operator.

Proof: Using (4.1.6) and Lemma (4.1.4);

If $m \leq f(x) \leq N$ then $m \leq T_{m,\alpha}(f; x) \leq N$ for $\varkappa \in [0,1]$.

Especially, if m=0, we obtain;

If $f(\varkappa) \geq 0$ then $T_{m,\alpha}(f; \varkappa) \geq 0, \varkappa \in [0,1]$.

4.2 Approximation by the Generalized Bernstein Operator

In this section, for Blending Type Bernstein operators we investigate the degree of approximation by using the Lipschitz class and modulus of continuity.

Theorem 4.2.1: The α-Bernstein operator has specific notation by means of difference operators. That is;

$$T_{m,\alpha}(f; \varkappa) = \sum_{s=0}^{m} \left[(1-\alpha)\binom{m-1}{s}\Delta^s g_0 + \alpha\binom{m}{s}\Delta^s f_0\right]\varkappa^s$$

where;

$$f_j = f\left(\frac{j}{m}\right), \qquad \lambda_j = \left(1 - \frac{j}{m-1}\right)f_j + \frac{j}{m-1}f_{j+1}$$

Proof: Expand the term (4.1.3) with $(1-x)^{m-j-1}$, we have

$$k_1 + k_2 = \sum_{j=1}^{m-1} g_j \binom{m-1}{j} \sum_{\ell=0}^{m-\ell-1} (-1)^\ell \binom{m-j-1}{\ell} x^\ell$$

Put $j + \ell$ instead of s, then we have;

$$\sum_{j=0}^{m-1}\sum_{\ell=0}^{m-j-1} = \sum_{s=0}^{m-1}\sum_{j=0}^{m},$$

and also we can write;

$$\binom{m-1}{j}\binom{m-j-1}{\ell} = \binom{s}{j}\binom{m-1}{s},$$

then put the double summation as;

$$k_1 + k_2 = \sum_{j=1}^{m-1}\binom{m-1}{s}\varkappa^s\sum_{\ell=0}^{s}(-1)^{s-j}\binom{s}{j}\lambda_j.$$

When we use the expansion for difference formula;

$$k_1 + k_2 = \sum_{r=0}^{m}\binom{m-1}{s}\varkappa^s\Delta^s\lambda_0.$$

When s = m, the sum is zero.

$$\sum_{j=1}^{m} f_j\binom{m}{j}\varkappa^j(1-\varkappa)^{m-j} = \sum_{s=0}^{m}\binom{m}{s}\varkappa^s\Delta^s f_0 .$$

So we get;

$$T_{m,\alpha}(f;\varkappa) = \sum_{s=0}^{m}\left[(1-\alpha)\binom{m-1}{s}\Delta^s g_0 + \alpha\binom{m}{s}\Delta^s f_0\right]\varkappa^s .$$

Lemma 4.2.1: We can express the higher order difference of λ_j as follows;

$$\Delta^s\lambda_j = \left(1-\frac{j}{m-1}\right)\Delta^s f_j + \frac{j+s}{m-1}\Delta^s f_{j+1} \tag{4.2.1}$$

Proof: it can be proved by induction.

i. When s=0, the equation is correct.

ii. Assume it holds for s = k − 1.

iii. Prove it for s= k.

$$\Delta^k\lambda_j = \Delta\left[\Delta^{k-1}\lambda_j\right]$$

$$\Delta^{k}\lambda_{j} = \Delta\left[\left(1-\frac{j}{m-1}\right)\Delta^{k-1}f_{j} + \frac{j+k-1}{m-1}\Delta^{k-1}f_{j+1}\right]$$

$= \left[\Delta^{k-1}f_{j}\Delta\left(1-\frac{j}{m-1}\right) + \left(1-\frac{j+1}{m-1}\right)\Delta\left(\Delta^{k-1}f_{j}\right)\right] + \left[\Delta^{k-1}f_{j+1}\Delta\left(\frac{j+k-1}{m-1}\right) + \right.$

$\left.\frac{j+1+k-1}{m-1}\Delta\left(\Delta^{k-1}f_{j+1}\right)\right]$

$= \left[\Delta^{k-1}f_{j}\Delta\left(1-\frac{j}{m-1}\right) + \left(1-\frac{j+1}{m-1}\right)\Delta^{k}f_{j}\right] + \left[\Delta^{k-1}f_{j+1}\Delta\left(\frac{m+k-1}{j-1}\right) + \frac{j+k}{m-1}\Delta^{k}f_{j+1}\right]$

$= \left[\Delta^{k-1}f_{j}\left(-\frac{1}{m-1}\right) + \left(1-\frac{j+1}{m-1}\right)\Delta^{k}f_{j}\right] + \left[\Delta^{k-1}f_{j+1}\left(\frac{1}{m-1}\right) + \frac{j+k}{m-1}\Delta^{k}f_{j+1}\right]$

$= \left(1-\frac{j}{m-1}\right)\Delta^{k}f_{j} + \frac{j+k}{m-1}\Delta^{k}f_{j+1}.$

From the differences and derivatives, we have;

$$\Delta^{s}f_{0} = 0 \text{ for } s > k$$

and

$$\Delta^{k}f_{j} = \frac{k!}{m^{k}}.$$

From Lemma (4.2.2) with $f(x)=x^{k}$ and $m-1 \geq k$ that

$$\Delta^{s}\lambda_{0} = 0 \text{ for } s > k$$

and

$$\Delta^{k}\lambda_{0} = \Delta^{k}f_{0} + \frac{k}{m-1}\Delta^{k}f_{1} = \left(1+\frac{k}{m-1}\right)\frac{k!}{m^{k}}$$

Especially from $f(\varkappa)=\varkappa^{k}$ and $m-1 \geq k$ that

$$T_{m,\alpha}\left(\varkappa^{k};\varkappa\right) = a_{k}\varkappa^{k} + a_{k-1}\varkappa^{k-1} + \cdots + a_{1}\varkappa + a_{o}$$

where

$$a_{k} = (1-\alpha)\binom{m-1}{k}\Delta^{k}\lambda_{0} + \alpha\binom{m}{k}\Delta^{k}f_{0}.$$

Write in the place $\Delta^{k}\lambda_{0}$ and $\Delta^{k}f_{0}$, we have;

$$a_{k} = \left[(1-\alpha)\binom{m-1}{k}\left(1+\frac{k}{m-1}\right) + \alpha\binom{m}{k}\right]\frac{k!}{m^{k}}.$$

For k is 0 and 1, a_{k} is 1. Then ;

$$a_k = \left[1 + \frac{k(k-1)}{m(m-1)}(\alpha - 1)\right]\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\dots\left(1 - \frac{k-1}{m}\right), \qquad \text{for } k \geq 2.$$

Lemma 4.2.2: Following identities hold for α −Bernstein operators.

i. $T_{m,\alpha}(k^2;\varkappa) = \varkappa^2 + \frac{m+2(1-\alpha)}{m^2}\varkappa(1-\varkappa)$

ii. $T_{m,\alpha}(k^3;\varkappa) = \varkappa^3 + \frac{3[m+2(1-\alpha)]}{m^2}\varkappa^2(1-\varkappa) + \frac{m+6(1-\alpha)}{m^3}\varkappa(1-\varkappa)(1-2\varkappa)$

iii. $T_{m,\alpha}(k^4;\varkappa) = \varkappa^4 + \frac{6[m+2(1-\alpha)]}{m^2}\varkappa^3(1-\varkappa) + \frac{4[m+6(1-\alpha)]}{m^3}\varkappa^2(1-\varkappa)(1-$

$2\varkappa) + \frac{[3m(m-2)+12(m-6)(1-\alpha)]\varkappa(1-\varkappa)+[m+14(1-\alpha)]}{m^4}\varkappa(1-\varkappa)$

Proof:

i. We know;

$$T_{m,\alpha}(\varkappa^k;\varkappa) = a_k\varkappa^k + a_{k-1}\varkappa^{k-1} + \dots + a_1\varkappa + a_o$$

where

$$a_k = (1-\alpha)\binom{m-1}{k}\Delta^k\lambda_0 + \alpha\binom{m}{k}\Delta^k f_0$$

So,

$$T_{m,\alpha}(\varkappa^2;\varkappa) = a_2\varkappa^2 + a_1\varkappa + a_o$$

$= \left[(1-\alpha)\binom{m-1}{2}\Delta^2\lambda_0 + \alpha\binom{m}{2}\Delta^2 f_0\right]\varkappa^2 + \left[(1-\alpha)\binom{m-1}{1}\Delta^1\lambda_0 + \right.$

$\left.\alpha\binom{m}{1}\Delta^1 f_0\right]\varkappa + \left[(1-\alpha)\binom{m-1}{0}\Delta^0\lambda_0 + \alpha\binom{m}{0}\Delta^0 f_0\right]$

$= \left[(1-\alpha)\binom{m-1}{2}\frac{2}{m^2}\left(\frac{m+1}{m-1}\right) + \alpha\binom{m}{2}\frac{2}{m^2}\right]\varkappa^2 + \left[(1-\alpha)\binom{m-1}{1}\frac{1}{m^2}\left(\frac{m+2}{m-1}\right) + \right.$

$\left.\alpha\binom{m}{1}\frac{1}{m^2}\right]\varkappa$

where

$$\Delta^0 f_0 = 0, \qquad \Delta f_0 = \frac{1}{m^2}, \qquad \Delta^2 f_0 = \frac{2}{m^2}$$

And

$$\Delta^0\lambda_0 = 0, \qquad \Delta\lambda_0 = \frac{1}{m^2}\left(\frac{m+2}{m-1}\right), \qquad \Delta^2\lambda_0 = \frac{2}{m^2}\left(\frac{m+1}{m-1}\right).$$

Then;

$$= \left[(1-\alpha)\frac{(m+1)(m-2)}{m^2} + \alpha\frac{m(m-1)}{m^2}\right]\varkappa^2 + \left[(1-\alpha)\frac{(m+2)}{m^2} + \alpha\frac{m}{m^2}\right]\varkappa$$

$$= \left[\frac{m^2-m-2(1-\alpha)}{m^2}\right]\varkappa^2 + \left[\frac{m+2(1-\alpha)}{m^2}\right]\varkappa$$

$$= \varkappa^2 + \frac{m+2(1-\alpha)}{m^2}\varkappa(1-\varkappa).$$

We can similarly proof $T_{m,\alpha}(k^3;\varkappa)$ and $T_{m,\alpha}(k^4;\varkappa)$.

Theorem 4.2.2: The α-Bernstein operators for $\alpha \in [0,1]$, converge uniformly to $f(\varkappa)$ that is a continuous function in [0,1].

Proof: Using lemma (4.1.2) and (4.2.3), this theorem can be easily proved.

Lemma 4.2.3: Let

$$K_s(\varkappa) = \sum_{j=0}^{m}(j - m\varkappa)^s P_{m,j}^{\alpha}(\varkappa), \qquad s = 0,1,2,3,4.$$

Then followings hold for $K_s(x)$:

i. $K_1(\varkappa) = 0.$

ii. $K_2(\varkappa) = [m + 2(1-\alpha)]\varkappa(1-\varkappa)$

iii. $K_3(\varkappa) = [m + 6(1-\alpha)]\varkappa(1-\varkappa)(1-2\varkappa)$

iv. $K_4(x) = \{[3m(m-2) + 12(m-6)(1-\alpha)]\varkappa(1-\varkappa) + m + 14(1-\alpha)\}\varkappa(1-\varkappa)$

Proof: Let's use binomial expansion of $(j - m\varkappa)^s$, s=0,1,2,3,4 and Lemma (4.1.2) and (4.2.3) we have;

$$\sum_{j=1}^{m} P_{m,j}^{\alpha}(\varkappa) = 1$$

$$\sum_{j=1}^{m} jP_{m,j}^{\alpha}(\varkappa) = m\varkappa$$

$$\sum_{j=1}^{m} j^2P_{m,j}^{\alpha}(\varkappa) = m^2\varkappa^2 + [m + 2(1-\alpha)]\varkappa(1-\varkappa)$$

$$\sum_{j=1}^{m} j^3P_{m,j}^{\alpha}(\varkappa) = m^3\varkappa^3 + 3m[m + 2(1-\alpha)]\varkappa^2(1-\varkappa) + [m + 6(1-\alpha)]\varkappa(1-\varkappa)(1 - 2\varkappa)$$

$$\sum_{j=1}^{m} j^4P_{m,j}^{\alpha}(\varkappa) = m^4\varkappa^4 + 6m^2[m + 2(1-\alpha)]\varkappa^3(1-\varkappa) + 4m[m + 6(1-\alpha)]\varkappa^2(1-\varkappa)(1-2\varkappa) + \{[3m(m-2) + 12(m-6)(1-\alpha)]\varkappa(1-\varkappa) + m + 14(1-\alpha)\}\varkappa(1-\varkappa)$$

Then;

$$K_1(\varkappa) = \sum_{j=0}^{m} (j - m\varkappa)^1 P_{m,j}^{\alpha}(\varkappa)$$

$$= jP_{m,j}^{\alpha}(\varkappa) - m\varkappa P_{m,j}^{\alpha}(\varkappa)$$

$$= m\varkappa - m\varkappa$$

$$= 0 \qquad (4.2.2)$$

$$K_2(\varkappa) = \sum_{j=0}^{m} (j - m\varkappa)^2 P_{m,j}^{\alpha}(\varkappa)$$

$$= \sum_{j=0}^{m} (j^2 - 2m\varkappa j + m^2\varkappa^2) P_{m,j}^{\alpha}(\varkappa)$$

$$= \sum_{j=0}^{m} j^2 P_{m,j}^{\alpha}(\varkappa) - 2m\varkappa\, jP_{m,j}^{\alpha}(\varkappa) + m^2\varkappa^2 P_{m,j}^{\alpha}(\varkappa)$$

$$= \sum_{j=0}^{m} m^2\varkappa^2 + [m + 2(1-\alpha)]\varkappa(1-\varkappa) - 2m^2\varkappa^2 + m^2\varkappa^2$$

$$= [m + 2(1-\alpha)]\varkappa(1-\varkappa) \tag{4.2.3}$$

$$K_3(\varkappa) = \sum_{j=0}^{m} (j - m\varkappa)^3 P_{m,j}^{\alpha}(\varkappa)$$

$$= \sum_{j=0}^{m} (j^3 - 3j^2 m\varkappa + 3jm^2\varkappa^2 - m^3\varkappa^3) P_{m,j}^{\alpha}(\varkappa)$$

$$= \sum_{j=0}^{j} j^3 P_{m,j}^{\alpha}(\varkappa) - 3m\varkappa j^2 P_{m,j}^{\alpha}(\varkappa) + 3m^2\varkappa^2 j P_{m,j}^{\alpha}(\varkappa) - m^3\varkappa^3 P_{m,j}^{\alpha}(\varkappa)$$

$$= m^3\varkappa^3 + 3m[m + 2(1-\alpha)]\varkappa^2(1-\varkappa) + [m + 6(1-\alpha)]\varkappa(1-\varkappa)(1 -$$

$$2\varkappa) - 3m\varkappa\{m^2\varkappa^2 + [m + 2(1-\alpha)]\varkappa(1-\varkappa)\} + 3m^3\varkappa^3 - m^3\varkappa^3$$

$$= m^3\varkappa^3 + 3m\varkappa^2[m + 2(1-\alpha)](1-\varkappa) + [m + 6(1-\alpha)]\varkappa(1-\varkappa)(1 -$$

$$2\varkappa) - 3m^3\varkappa^3 - 3m\varkappa^2[m + 2(1-\alpha)](1-\varkappa) + 3m^3\varkappa^3 - m^3\varkappa^3$$

$$= [m + 6(1-\alpha)]\varkappa(1-\varkappa)(1-2\varkappa) \tag{4.2.4}$$

$$K_4(\varkappa) = \sum_{j=0}^{m} (j - m\varkappa)^4 P_{m,j}^{\alpha}(\varkappa)$$

$$= \left(j^4 - 4j^3 m\varkappa + 6j^2 m^2\varkappa^2 + 4j^2 m^2\varkappa^2 + m^4\varkappa^4\right) P_{m,j}^{\alpha}(\varkappa)$$

$$= j^4 P_{m,j}^{\alpha}(\varkappa) - 4m\varkappa j^3 P_{m,j}^{\alpha}(\varkappa) + 6m^2\varkappa^2 j^2 P_{m,j}^{\alpha}(\varkappa) - 4m^3\varkappa^3 j P_{m,j}^{\alpha}(\varkappa)$$

$$+ m^4 x^4 P_{m,j}^{\alpha}(\varkappa)$$

$$= m^4\varkappa^4 + 6m^2[m + 2(1-\alpha)]\varkappa^3(1-\varkappa) + 4m[m + 6(1-\alpha)]\varkappa^2(1 -$$

$$-\varkappa)(1-2\varkappa) + \{[3m(m-2) + 12(m-6)(1-\alpha)]\varkappa(1-\varkappa) + m$$

$$+14(1-\alpha)\}\varkappa(1-\varkappa) - 4m\varkappa\{m^3\varkappa^3 + 3m[m + 2(1-\alpha)]\varkappa^2(1-\varkappa)$$

$$+[m + 6(1-\alpha)]\varkappa(1-\varkappa)(1-2\varkappa) + 6m^2\varkappa^2\{m^2\varkappa^2 + [m + 2(1-\alpha)]$$

$$\varkappa(1-\varkappa)\} - 4m^4\varkappa^4 + m^4\varkappa^4$$

$$= \{[3m(m-2) + 12(m-6)(1-\alpha)]\varkappa(1-\varkappa) + m + 14(1-\alpha)\}\varkappa(1-\varkappa)$$

(4.2.5)

Lemma 4.2.4: ∃ a constant G that is independent on m such that ∀ $\varkappa$ on [0,1] and any real $\delta \in \left(0, \frac{1}{4}\right)$,

$$\sum_{\left|\frac{j}{m}-\varkappa\right| \geq m^{-\delta}} P_{m,j}^{\alpha}(\varkappa) \leq G m^{2(2\delta-1)}.$$

Proof: From Lemma (4.2.4) for constant G, $|S_4(\varkappa)| \leq G m^2$.

Seeing that;

$$\left|\frac{j}{m} - \varkappa\right| \geq m^{-\delta}$$

$$\frac{(j - m\varkappa)^4}{m^4} \geq m^{-4\delta}$$

$$(j - m\varkappa)^4 \geq m^{4(1-\delta)}$$

$$(j - m\varkappa)^4 m^{4(\delta-1)} \geq 1.$$

Then we have,

$$\sum_{\left|\frac{j}{m}-\varkappa\right| \geq m^{-\delta}} P_{m,j}^{\alpha}(\varkappa) \leq m^{4(\delta-1)} \sum_{j=0}^{m} (j - m\varkappa)^4 P_{m,j}^{\alpha}(\varkappa) = m^{4(\delta-1)} K_4(\varkappa) \leq G m^{2(2\delta-1)}.$$

Theorem 4.2.3: Assume $f(\varkappa)$ is bounded on the [0,1]. In any $\varkappa \in [0,1]$ where $f''(x)$ is defined at $0 \leq \alpha \leq 1$ as;

$$\lim_{m\to\infty} m\left[T_{m,\alpha}(f;\varkappa) - f(\varkappa)\right] = \frac{1}{2}\varkappa(1-\varkappa)f''(x)$$

Proof: For $j \leq m$, using Taylor's formula such that;

$$f(k) = f(\varkappa) + (k-\varkappa)f'(\varkappa) + \frac{1}{2}(k-\varkappa)^2 f''(\varkappa) + g(k)(k-\varkappa)^2$$

where $\lim_{k\to\varkappa} g(k) = 0$.

Take $k = \frac{j}{m}$, then;

$$f\left(\frac{j}{m}\right) = f(\varkappa) + \left(\frac{j}{m} - \varkappa\right)f'(x) + \frac{1}{2}\left(\frac{j}{m} - \varkappa\right)^2 f''(\varkappa) + g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2.$$

Then,

$$m\left[T_{m,\alpha}(f;\varkappa) - f(\varkappa)\right] = m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left[f\left(\frac{j}{m}\right) - f(\varkappa)\right]$$

$$= m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left[f(\varkappa) + \left(\frac{j}{m} - \varkappa\right)f'(x) + \frac{1}{2}\left(\frac{j}{m} - \varkappa\right)^2 f''(x) + g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2 \right.$$

$$\left. - f(\varkappa)\right]$$

$$= m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left[\left(\frac{j}{m} - \varkappa\right)f'(\varkappa) + \frac{1}{2}\left(\frac{j}{m} - \varkappa\right)^2 f''(\varkappa) + g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2\right]$$

$$= m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left(\frac{j}{m} - \varkappa\right)f'(\varkappa) + \frac{m}{2}\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left(\frac{j}{m} - \varkappa\right)^2$$

$$+ m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa) g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2$$

$$= K_1(\varkappa)f'(x) + \frac{1}{2m}K_2(\varkappa)f''(\varkappa) + m\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa) g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2$$

Then using (4.2.2) and (4.3.3) we get;

$$m\left[T_{m,\alpha}(f;\varkappa) - f(\varkappa)\right] = \left(\frac{1}{2} + \frac{1-\alpha}{m}\right)\varkappa(1-\varkappa)f''(x) + mS_m(\varkappa)$$

where

$$R_m(\varkappa) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right)\left(\frac{j}{m} - \varkappa\right)^2 P_{m,j}^{\alpha}(\varkappa).$$

For $0 \le \alpha \le 1$, we can get the following inequality;

$$|R_m(\varkappa)| \le \sum_{\left|\frac{j}{m} - \varkappa\right| \ge m^{-\frac{1}{8}}} \left|g\left(\frac{j}{m}\right)\right|\left(\frac{j}{m} - \varkappa\right)^2 P_{m,j}^{\alpha}(\varkappa) + \sum_{\left|\frac{j}{m} - \varkappa\right| \ge m^{-\frac{1}{8}}} \left|g\left(\frac{j}{m}\right)\right|\left(\frac{j}{m} - \varkappa\right)^2 P_{m,j}^{\alpha}(\varkappa).$$

Let's give any $\varepsilon > 0$. sufficiently large m can be found as $\left|\frac{j}{m} - \varkappa\right| \geq m^{-\frac{1}{8}}$ implies $\left|g\left(\frac{j}{m}\right)\right| < \varepsilon$.

So that,

$$|R_m(\varkappa)| \leq \frac{\varepsilon}{m^2} K_2(\varkappa) + M \sum_{\left|\frac{j}{m} - \varkappa\right| \geq m^{-\frac{1}{8}}} P_{m,j}^{\alpha}(\varkappa)$$

where $M = \sup_{0 \leq k \leq 1} g(k)(k - \varkappa)^2$. From lemma (4.2.5) for $\delta = \frac{1}{8}$;

$$m|R_m(\varkappa)| \leq \varepsilon\left[1 + \frac{2(1-\alpha)}{m}\right]\varkappa(1-\varkappa) + \frac{MG}{m^{\frac{1}{2}}} \; .$$

This completes the proof as ε is arbitrary.

Theorem 4.2.4: If f is bounded on [0,1], and $0 \leq \alpha \leq 1$, then

$$\left\|f(\varkappa) - T_{m,\alpha}(f;\varkappa)\right\| \leq \frac{3}{2}\omega\left(\frac{\sqrt{m + 2(1-\alpha)}}{m}\right).$$

Proof:

For $0 \leq \alpha \leq 1$,

$$\left|f(\varkappa) - T_{m,\alpha}(f;\varkappa)\right| = \left|\sum_{j=0}^{m} P_{m,j}^{\alpha}(\varkappa)\left[f(\varkappa) - f\left(\frac{j}{m}\right)\right]\right|$$

$$\leq \sum_{j=0}^{m}\left|f(\varkappa) - f\left(\frac{j}{m}\right)\right| P_{m,j}^{\alpha}(\varkappa)$$

$$\leq \sum_{j=0}^{m}\omega\left(\left|\varkappa - \frac{j}{m}\right|\right) P_{m,j}^{\alpha}(\varkappa).$$

By the properties of modulus of continuity, we get;

$$\left(\left|\varkappa - \frac{j}{m}\right|\right) = \omega\left(\frac{m}{\sqrt{m + 2(1-\alpha)}}\left|\varkappa - \frac{j}{m}\right|\sqrt{\frac{m + 2(1-\alpha)}{m}}\right)$$

$$\leq \left(1+\frac{m}{\sqrt{m+2(1-\alpha)}}\left|\varkappa-\frac{j}{m}\right|\right)\omega\left(\sqrt{\frac{m+2(1-\alpha)}{m}}\right)$$

hence

$$\left|f(\varkappa)-T_{m,\alpha}(f;\varkappa)\right| \leq \sum_{j=0}^{m}\left(1+\frac{m}{\sqrt{m+2(1-\alpha)}}\left|\varkappa-\frac{j}{m}\right|\right)\omega\left(\sqrt{\frac{m+2(1-\alpha)}{m}}\right)P_{m,j}^{\alpha}(\varkappa)$$

$$\leq \omega\left(\sqrt{\frac{m+2(1-\alpha)}{m}}\right)\left(1+\frac{m}{\sqrt{m+2(1-\alpha)}}\sum_{j=0}^{m}\left|\varkappa-\frac{j}{m}\right|P_{m,j}^{\alpha}(\varkappa)\right).$$

Using the Cauchy-Schwarz's inequality;

$$\sum_{j=0}^{m}\left|\varkappa-\frac{j}{m}\right|P_{m,j}^{\alpha}(\varkappa)=\sum_{j=0}^{m}\left|\varkappa-\frac{j}{m}\right|\sqrt{P_{m,j}^{\alpha}(\varkappa)}\sqrt{P_{m,j}^{\alpha}(\varkappa)}$$

$$\leq \left[\sum_{j=0}^{m}\left(\varkappa-\frac{j}{m}\right)^{2}P_{m,j}^{\alpha}(\varkappa)\right]^{\frac{1}{2}}\left[\sum_{j=0}^{m}P_{m,j}^{\alpha}(\varkappa)\right]^{\frac{1}{2}}$$

$$=\left[\sum_{j=0}^{m}\left(\varkappa-\frac{j}{m}\right)^{2}P_{m,j}^{\alpha}(\varkappa)\right]^{\frac{1}{2}}.$$

Moreover,

$$\sum_{j=0}^{m}\left(\varkappa-\frac{j}{m}\right)^{2}P_{m,j}^{\alpha}(\varkappa)=\frac{1}{m^{2}}K_{2}(\varkappa)=\frac{m+2(1-\alpha)}{m^{2}}\varkappa(1-\varkappa)\leq\frac{m+2(1-\alpha)}{4m^{2}}.$$

In the seem of (4.2.3);

$$\left|f(\varkappa)-T_{m,\alpha}(f;\varkappa)\right| \leq \omega\left(\frac{\sqrt{m+2(1-\alpha)}}{m}\right)\left(1+\frac{m}{\sqrt{m+2(1-\alpha)}}.\frac{\sqrt{m+2(1-\alpha)}}{2m}\right)$$

$$\left|f(\varkappa)-T_{m,\alpha}(f;\varkappa)\right| = \frac{3}{2}\omega\left(\frac{\sqrt{m+2(1-\alpha)}}{m}\right).$$

4.3 Shape Preserving Properties

In the last section, we investigate the shape preserving properties by monotonicity and convex properties of α −Bernstein operators.

Theorem 4.3.1: Assume that ƒ∈C[0, 1]. If ƒ is monotonically increasing or decreasing on [0, 1] for $0 \leq \alpha \leq 1$ then it realizes samely for all α – Bernstein operators.

Proof:

We can write;

$$T_{m+1,\alpha}(f;\varkappa) = (1-\alpha)\left[\sum_{j=0}^{m+1} f_j \binom{m-1}{j} \varkappa^j (1-\varkappa)^{m-j} + \sum_{j=0}^{m+1} f_j \binom{m-1}{j-2} \varkappa^{j-1}(1-\varkappa)^{m-j+1}\right] + \alpha B_{m+1}(f;\varkappa)$$

$\forall\, m \geq 0$ where $f_j = \left(\frac{j}{m+1}\right)$.

Follow (4.1.5), then;

$$T_{m+1,\alpha}(f;\varkappa) = (1-\alpha)\left[\sum_{j=0}^{m} \lambda_j \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j} + \alpha \sum_{j=0}^{m+1} f_j \binom{m+1}{j} \varkappa^j (1-\varkappa)^{n-j+1}\right]$$

where $\lambda_j = \left(1-\frac{j}{m}\right) f_j + \frac{j}{m} f_{j+1}$.

Calculating the derivative of $T_{m+1,\alpha}(f;\varkappa)$, then it gives;

$$T'_{m+1,\alpha}(f;\varkappa) = (1-\alpha)D_1 + \alpha D_2$$

where

$$D_1 = \sum_{j=0}^{m} \lambda_j \binom{m}{j} [j\varkappa^{j-1}(1-\varkappa)^{m-j} - (m-j)\varkappa^j (1-\varkappa)^{m-j-1}$$

and

$$D_2 = \sum_{j=0}^{m+1} f_j \frac{d}{d\varkappa} P_{m+1,j}(\varkappa).$$

In other words D_1 can be written as follows;

$$D_1 = \sum_{j=1}^{m} j \binom{m}{j} \lambda_j \varkappa^{j-1}(1-\varkappa)^{m-j} - \sum_{j=0}^{m-1} (m-j) \binom{m}{j} \lambda_j \varkappa^j (1-\varkappa)^{m-j-1}.$$

$$= \sum_{j=0}^{m-1} (j+1) \binom{m}{j+1} \lambda_{j+1} \varkappa^j (1-\varkappa)^{m-j-1} - \sum_{j=0}^{m-1} (m-j) \binom{m}{j} \lambda_j \varkappa^j (1-\varkappa)^{m-j-1}$$

$$= \left[\sum_{j=0}^{m-1} (j+1) \binom{m}{j+1} \lambda_{j+1} - \sum_{j=0}^{m-1} (m-j) \binom{m}{j} \lambda_j \right] \varkappa^j (1-\varkappa)^{m-j-1}.$$

Insomuch as;

$$(j+1) \binom{m}{j+1} = m \binom{m-1}{j} = (m-j) \binom{m}{j},$$

we get

$$D_1 = \sum_{j=0}^{m-1} m \Delta \lambda_j \binom{m-1}{j} \varkappa^j (1-\varkappa)^{m-j-1}$$

where Δ is the forward difference operator. Subsequently;

$$D_1 = \sum_{j=0}^{m-1} \left[(j+1)\Delta f_{j+1} + (m-j)\Delta f_j\right] \binom{m-1}{j} \varkappa^j (1-\varkappa)^{m-j-1}.$$

Similarly if proved in D_2;

$$D_2 = (m+1) \sum_{j=1}^{m} \Delta f_j \binom{m}{j} \varkappa^j (1-\varkappa)^{m-j-1},$$

is obtained. Therefore $T'_{m+1,\alpha}(f;\varkappa)$ is given below;

$$T'_{m+1,\alpha}(f;\varkappa) = (1-\alpha) \sum_{j=0}^{m-1} [(j+1)\Delta f_{j+1} + (m-j)\Delta f_j] \binom{m-1}{j} \varkappa^j (1-\varkappa)^{m-j-1}$$

$$+\alpha\sum_{j=0}^{m}\Delta f_j(m+1)\binom{m}{j}\varkappa^j(1-\varkappa)^{m-j}.$$

We can determine the sign of derivative of $T_{m+1,\alpha}(f;\varkappa)$ by forward difference. If function f is monotonically increasing, f's forward differences are not negative. So $T'_{m+1,\alpha}(f;\varkappa)$ is not negative on [0,1]. Therefore it is monotonically increasing. This is also true for the opposite. Hence, this completes the proof.

Theorem 4.3.2: Assume that $f \in C[0,1]$. α −Bernstein operators to be convex for $0 \leq \alpha \leq 1$ if $f(\varkappa)$ is convex on [0,1].

Proof:

Follow (4.1.5), then;

$$T_{m+2,\alpha}(f;\varkappa) = (1-\alpha)\sum_{j=0}^{m+1}\lambda_j\binom{m+1}{j}\varkappa^j(1-\varkappa)^{m+1-j}$$

$$+\alpha\sum_{j=0}^{m+2}f_j\binom{m+2}{j}\varkappa^j(1-\varkappa)^{m+2-j}$$

where $f_j = f\left(\frac{j}{m+2}\right)$ and $\lambda_j = \left(1-\frac{j}{m}\right)f_j + \frac{j}{m}f_{j+1}$.

Calculating the first derivative of Theorem (4.3.1), we get;

$$T''_{m+2,\alpha}(f;\varkappa) = (1-\alpha)m(m+1)\sum_{s=0}^{m-1}\Delta^2\lambda_s\binom{m-1}{s}\varkappa^s(1-\varkappa)^{m-s-1}$$

$$+\alpha(m+1)(m+2)\sum_{s=0}^{m}\Delta^2 f_s\binom{m}{s}\varkappa^s(1-\varkappa)^{m-s}.$$

Replace m by m+2, hence;

$$\Delta^2\lambda_s = \left(1-\frac{s}{m+1}\right)\Delta^2 f_s + \frac{s+2}{m+1}\Delta^2 f_{s+1}$$

$$\Delta^2 f_s = \frac{2}{(m+2)^2} f\left[\frac{s}{m+2}, \frac{s+1}{m+2}, \frac{s+2}{m+2}\right]$$

$$T''_{m+2,\alpha}(f;\varkappa) \geq 0.$$

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